Lecture 3 - Transverse Dynamics I

ACCELERATOR PHYSICS

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Slides for study before the lecture

Please study the slides on relativity and cyclotron focusing before the lecture and ask questions to clarify any points not understood

Relativistic definitions

Energy of a particle at rest

$$E_0 = m_0 c^2$$

Total energy of a moving particle (definition of γ)

$$E = \gamma E_0 = m_0 c^2 \gamma$$

$$\gamma = \frac{E}{E_0}$$

Another relativistic variable is defined:

$$\beta = \frac{\text{momentum} \times c}{\text{energy}} = \frac{pc}{E} = \frac{v}{c}$$

Alternative axioms you may have learned

$$E = \frac{m_0 c^2}{\sqrt{1-\beta^2}}$$

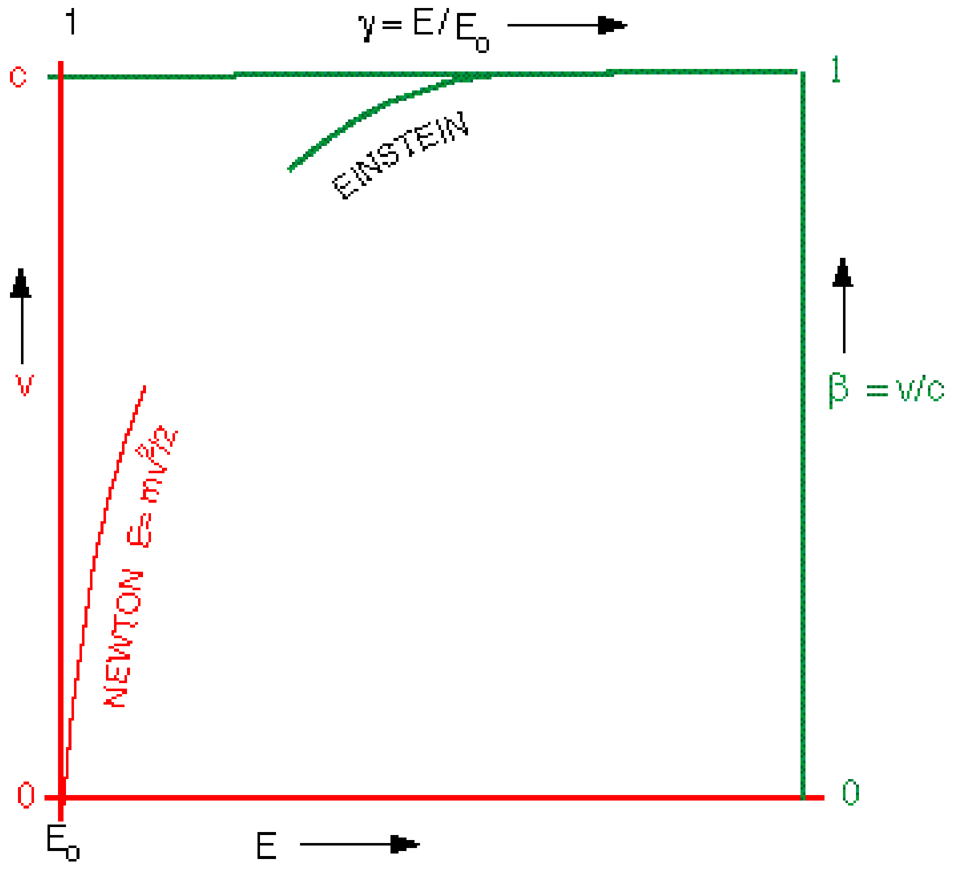
$$p = mv = \frac{m_0 v}{\sqrt{1-\beta^2}} = \frac{m_0 c \beta}{\sqrt{1-\beta^2}}$$

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\beta^2}}$$

You can prove:

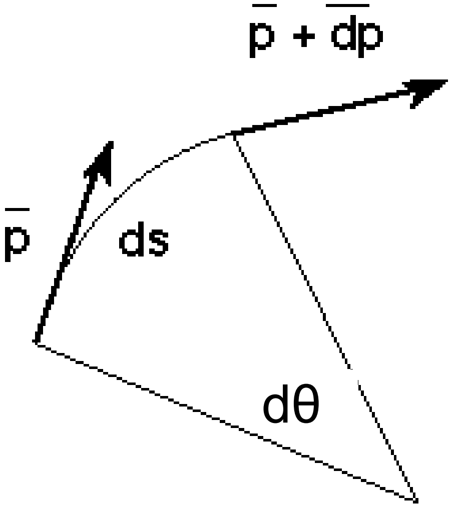
$$pc = \beta E = m_0 c^2 (\beta\gamma)$$

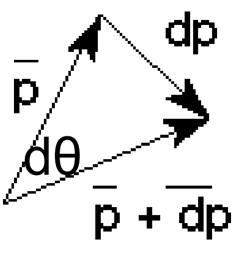
Newton & Einstein



- Almost all modern accelerators accelerate particles to speeds very close to that of light.
- In the classical Newton regime the velocity of the particle increases with the square root of the kinetic energy.
- As v approaches c it is as if the velocity of the particle "saturates"
- One can pour more and more energy into the particle, giving it a shorter De Broglie wavelength so that it probes deeper into the sub-atomic world
- Velocity increases very slowly and asymptotically to that of light

Magnetic rigidity





$$\frac{1}{\rho} = \frac{d\theta}{ds}$$

$$\frac{d\mathbf{p}}{dt} = |\mathbf{p}|\frac{d\theta}{dt} = |\mathbf{p}|\frac{d\theta}{ds}\frac{ds}{dt} = \frac{|\mathbf{p}|}{\rho}\frac{ds}{dt}$$

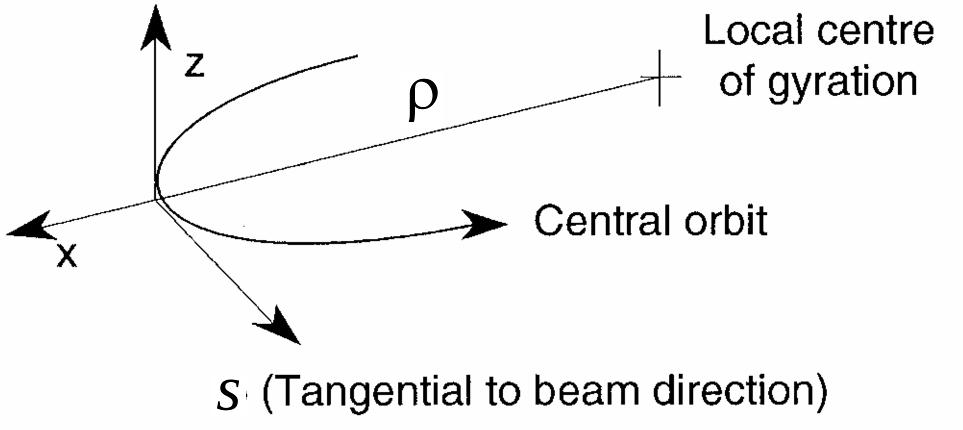
$$= e\mathbf{v} \times \mathbf{B} = e\frac{ds}{dt}B$$

$$(B\rho) = \frac{p}{e} = \frac{pc}{ec} = \frac{\beta E}{ec} = \frac{\beta\gamma E_0}{ec} = \frac{m_0 c}{e}(\beta\gamma)$$

$$(B\rho)\ [\mathrm{T.m}] = \frac{pc}{ec} = \frac{pc[eV]}{c[m.s^{-1}]} = 3.3356\ (\mathrm{pc})\ [\mathrm{GeV}]$$

Fig.Brho 4.8

Transverse coordinates



Equation of motion in a cyclotron

- **Non relativistic**

$$\frac{d(m\mathbf{v})}{dt} = \mathbf{F} \qquad\qquad \frac{d(m\mathbf{v})}{dt} = q\left[\mathbf{v}\times\mathbf{B}\right]$$

- **Cartesian**

$$\frac{d(mv_x)}{dt} = \frac{d(m\dot{x})}{dt} = q\left[\dot{y}B_z - \dot{z}B_y\right]$$

$$\frac{d(mv_y)}{dt} = \frac{d(m\dot{y})}{dt} = q\left[\dot{z}B_x - \dot{x}B_z\right]$$

$$\frac{d(mv_z)}{dt} = \frac{d(m\dot{z})}{dt} = q\left[\dot{x}B_y - \dot{y}B_x\right]$$

- **Cylindrical**

$$\frac{d(m\dot{r})}{dt} - mr\dot{\theta}^2 = q\left[r\dot{\theta}B_z - \dot{z}B_\theta\right]$$

$$\frac{d(mr\dot{\theta})}{dt} + m\dot{r}\dot{\theta} = q\left[\dot{z}B_r - \dot{r}B_z\right]$$

$$\frac{d(m\dot{z})^2}{dt} = q\left[rB_\theta - r\dot{\theta}B_r\right]$$

$w = \frac{q}{m_0} B_0$

Cyclotron orbit equation

- **For non-relativistic particles (m = m_0) and with an axial field B_z = -B_0**

$$m_0\left(\ddot{r} - r\dot{\theta}^2\right) = -qr\dot{\theta}B_z$$

$$m_0\left(r\ddot{\theta} + 2r\dot{\theta}\right) = q\dot{r}B_z$$

$$m_0\ddot{z} = 0$$

- **The solution is a closed circular trajectory which has radius**

$$R = \frac{p}{qB_z}$$

- **and an angular frequency**

$$\omega = \frac{q}{m_0} B_z$$

- **Take into account special relativity by**

$$m = m_0\gamma = m_0 \frac{E}{E_0}$$

- **And increase B with γ to stay synchronous!**

Cyclotron focusing – small deviations

- **See earlier equation of motion**

$$\frac{d\left(m\dot{r}\right)}{dt}+mr\dot{\theta}^{2}+q\left[r\dot{\theta}B_{z}-\dot{z}B_{\theta}\right]=0$$

- **If all particles have the same velocity:**

$$\rho\dot{\theta} = v_0 = \dot{z}$$

$$\frac{d}{dt}\left(m\frac{d\rho}{dt}\right)+\frac{mv_0{}^{2}}{\rho}+ev_0B_z=0$$

- **Change independent variable and substitute for small deviations**

$$\frac{d}{dt}=v_0\frac{d}{ds}\ , \qquad \Delta B_z=B_z-B_0, \qquad \mathrm{x}=\rho-\rho_0$$

- **Substitute**

$$p_0=mv_0$$

- **To give**

$$\frac{1}{mv_0}\frac{d}{ds}\left(p_0\frac{dx}{ds}\right)+\frac{x}{\rho_0{}^{2}}+\frac{1}{\rho_0}\frac{\Delta B_z}{B_0}=0$$

Cyclotron focusing – field gradient

- **From previous slide**

$$\frac{1}{mv_0}\frac{d}{ds}\left(p_0\frac{dx}{ds}\right)+\frac{x}{\rho_0{}^2}+\frac{1}{\rho_0}\frac{\Delta B_z}{B_0}=0$$

- **Taylor expansion of field about orbit**

$$B_z=B_0+\frac{\partial B_z}{\partial x}x+\frac{1}{2!}\frac{\partial^2 B_z}{\partial x^2}x^2+........$$

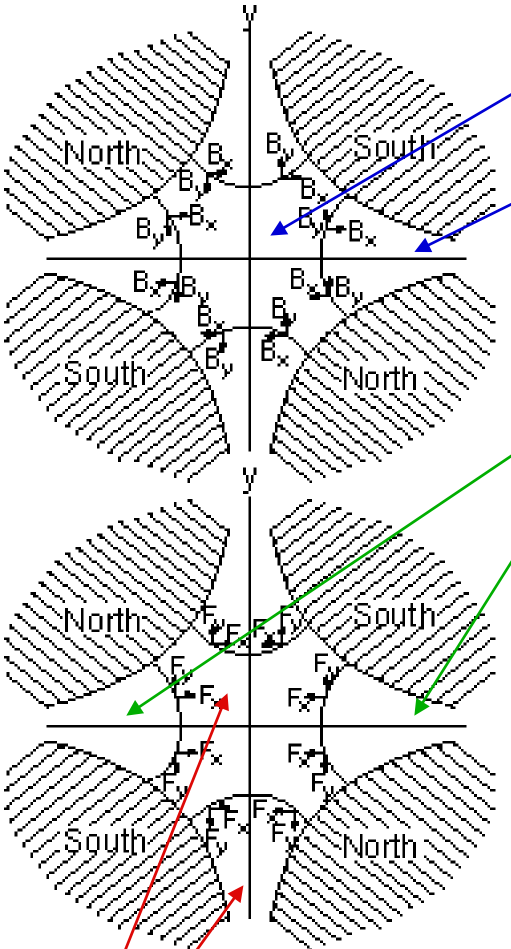
- **Define field index (focusing gradient)**

$$k=-\frac{1}{(B_0\rho_0)}\frac{\partial B_z}{\partial x}$$

- **To give horizontal focusing**

$$\frac{1}{p_0}\frac{d}{ds}\left(p_0\frac{dx}{ds}\right)+\left(\frac{1}{\rho^2}-k\right)x=0$$

Fields and force in a quadrupole



No field on the axis

Field strongest here

$$B \propto x$$

(hence is linear)

Force restores

Gradient $\rightarrow$ $\dfrac{\partial B_y}{\partial x}$

Normalised:

$$k = -\frac{1}{(B\rho)} \cdot \frac{\partial B_y}{\partial x}$$

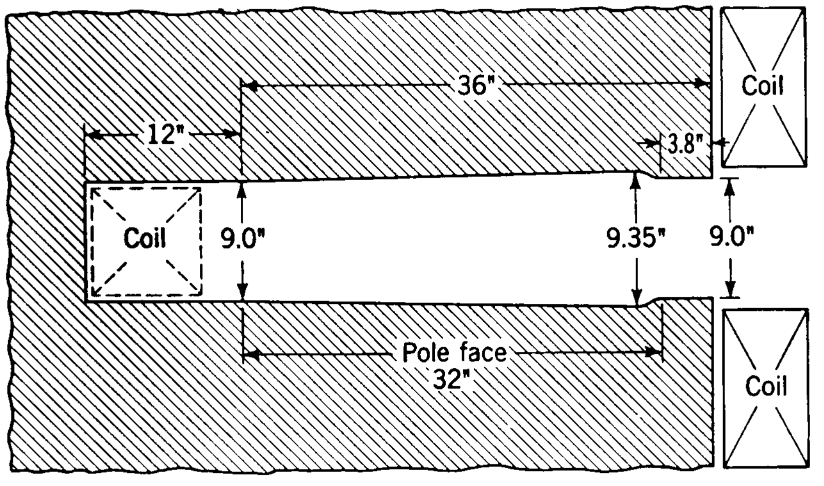
POWER OF LENS

$$\ell k = -\frac{\ell}{(B\rho)} \cdot \frac{\partial B_y}{\partial x} = \frac{1}{f}$$

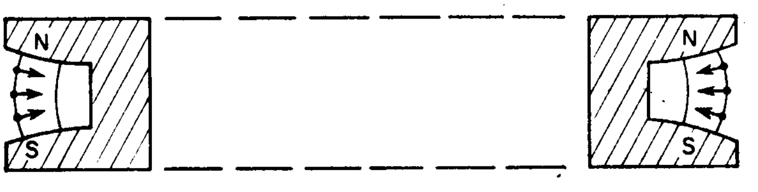
Defocuses in vertical plane

Fig. cas 10.8

Weak focusing in a synchrotron

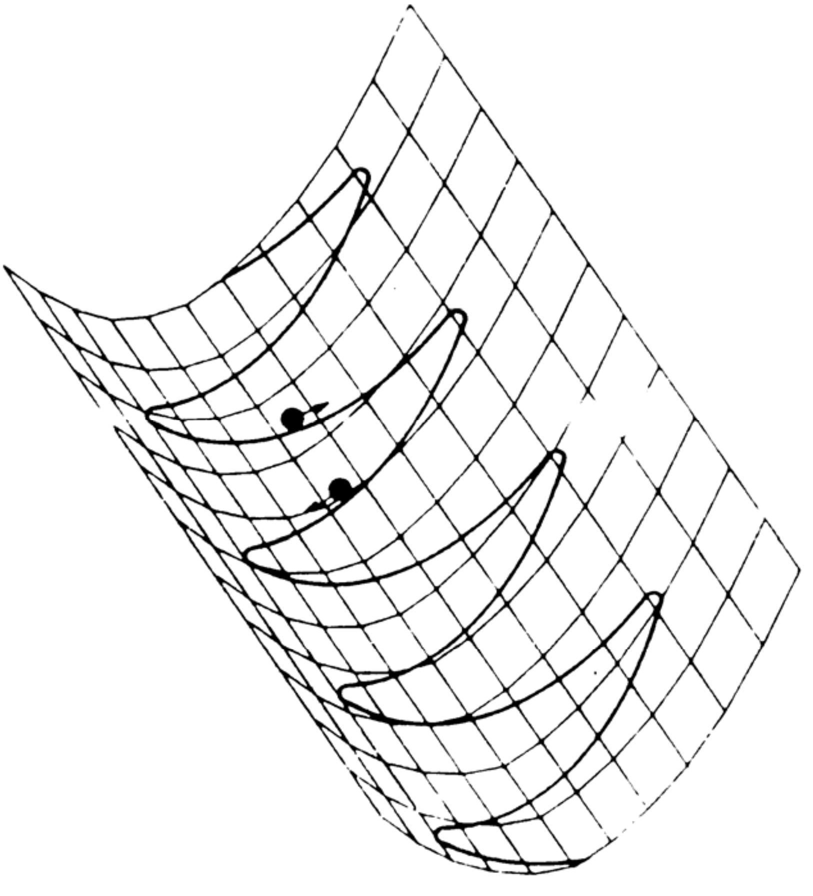


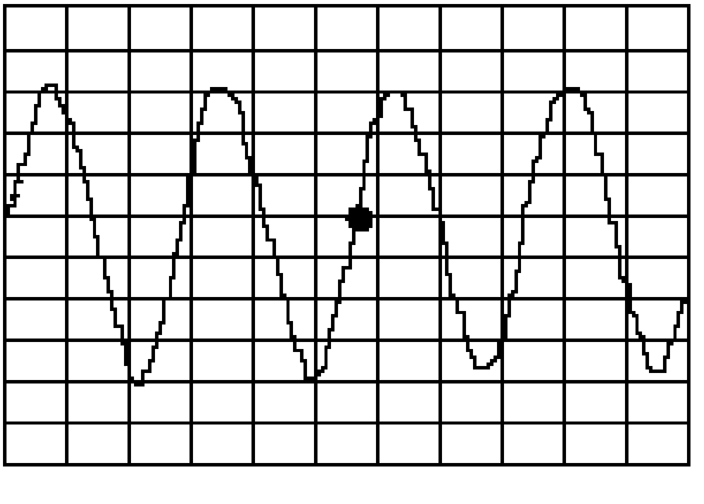
The Cosmotron magnet



- **Vertical focusing comes from the curvature of the field lines when the field falls off with radius (positive n-value)**
- **Horizontal focusing from the curvature of the path**
- **The negative field gradient defocuses horizontally and must not be so strong as to cancel the path curvature effect**

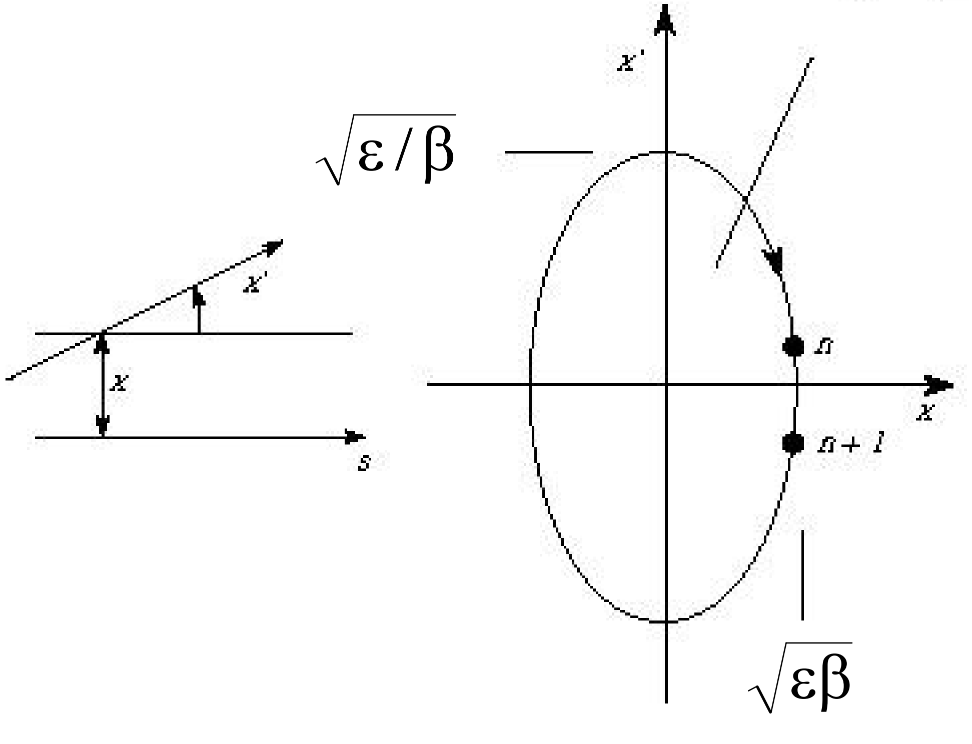
Gutter



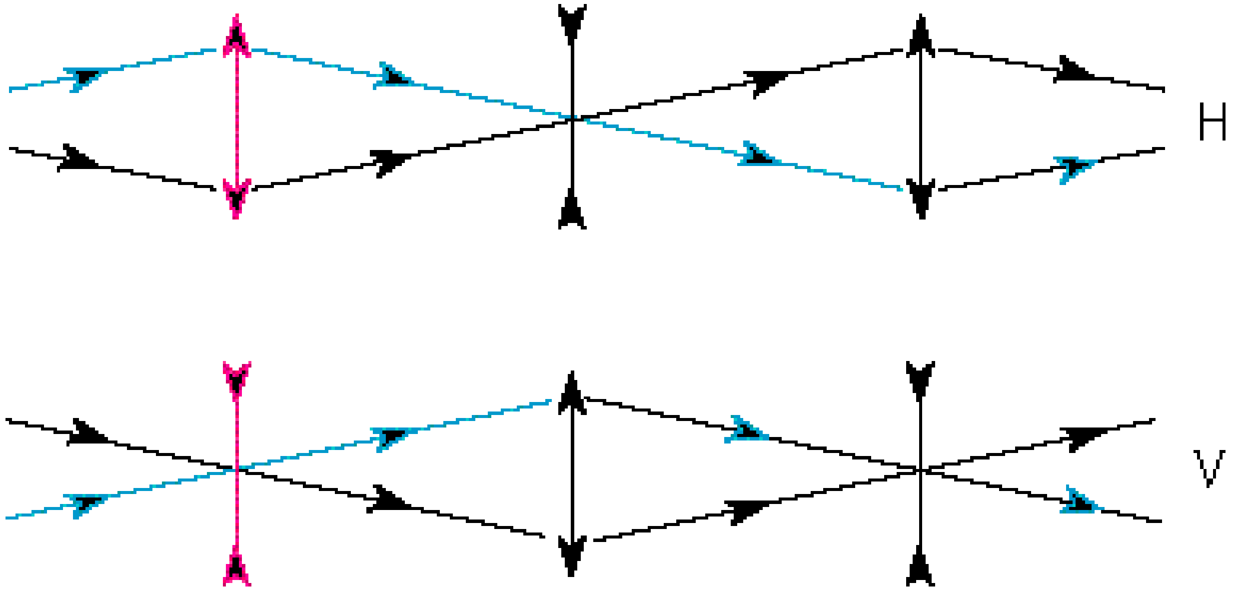


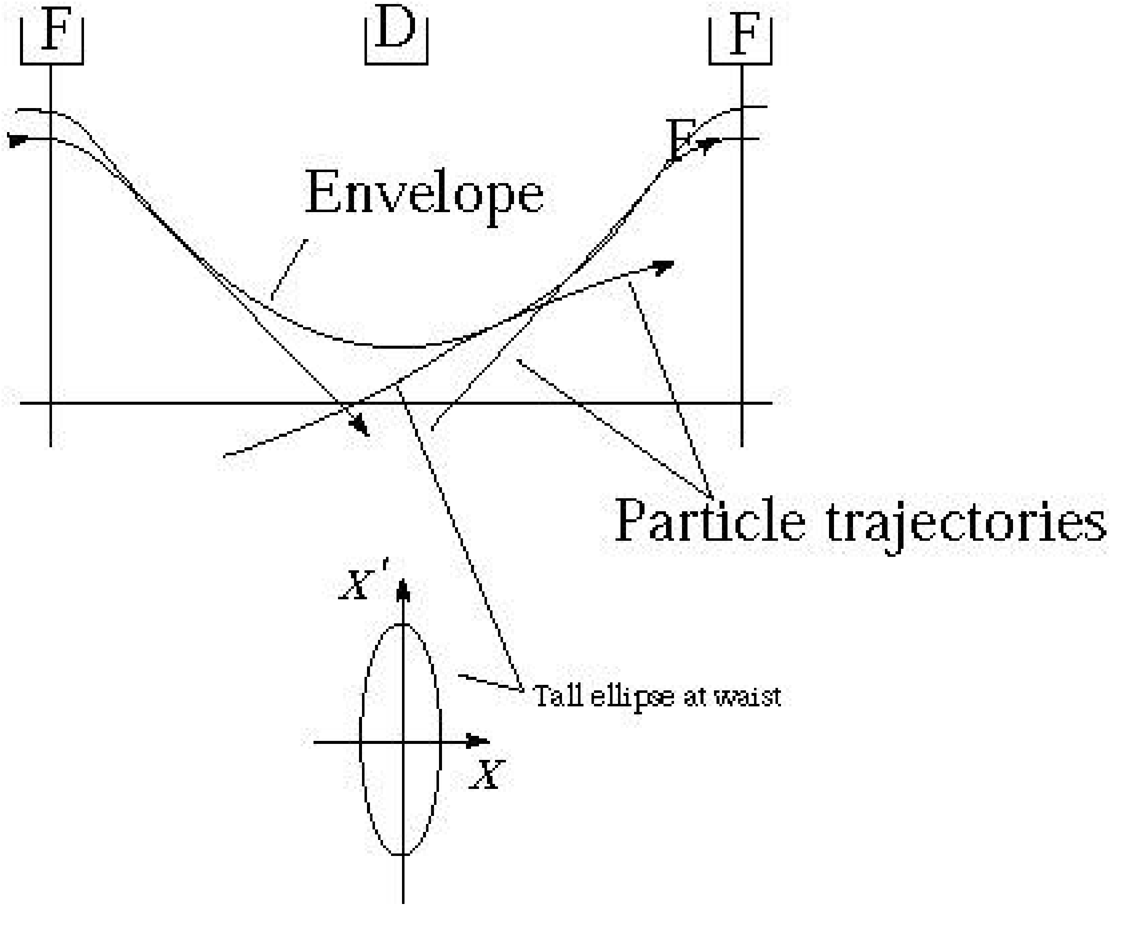
Transverse ellipse

$$\text{Area} = \pi\sqrt{\varepsilon\beta}.\sqrt{\varepsilon/\beta} = \pi\varepsilon$$



Alternating gradients





Equation of motion in transverse co-ordinates

- **Hill's equation (linear-periodic coefficients)**

$$\frac{d^2y}{ds^2} + k(s)y = 0$$

where $k = -\frac{1}{(B\rho)}\frac{dB_z}{dx}$ **at quadrupoles**

like restoring constant in harmonic motion

- **Solution (e.g. Horizontal plane)**

$$y = \sqrt{\beta(s)}\sqrt{\varepsilon}\sin\left[\phi(s) + \phi_0\right]$$

- **Condition**

$$\varphi = \int \frac{ds}{\beta(s)}$$

- **Property of machine** $\sqrt{\beta(s)}$
- **Property of the particle (beam) ε**
- **Physical meaning (H or V planes)**

Envelope $\sqrt{\varepsilon\beta(s)}$

Maximum excursions

$$\hat{y} = \sqrt{\varepsilon\beta(s)} \qquad \hat{y}' = \sqrt{\varepsilon / \beta(s)}$$

Check Solution of Hill

- **Differentiate** $y = \sqrt{\beta(s)\varepsilon}\cos\left(\phi(s) + \phi_o\right)$

  substituting $w = \sqrt{\beta}\ , \qquad \phi = \phi(s) + \phi_o$

$$y' = \varepsilon^{1/2}\left\{ w'(s)\cos\phi - \frac{d\phi}{ds} w(s)\sin\phi \right\}$$

- **Necessary condition for solution to be true**

$$\frac{d\phi}{ds} = \frac{1}{\beta(s)} = \frac{1}{w^2(s)}$$

so

$$y' = \varepsilon^{1/2}\left\{ w'(s)\cos\phi - \frac{1}{w(s)}\sin\phi \right\}$$

- **Differentiate again**

$$y'' = \varepsilon^{1/2}\left\{ w''(s)\cos\phi - \frac{w'(s)}{w^2(s)}\sin\phi + \frac{w'(s)}{w^2(s)}\sin\phi \right.$$

add both sides $\qquad - \frac{1}{w^3(s)}\cos\phi$

$+ky \qquad + kw(s)\cos\phi$

cancels to 0

must be zero

Continue checking

$$y'' = \varepsilon^{1/2}\left\{ w''(s)\cos\phi - \frac{w'(s)}{w^2(s)}\sin\phi + \frac{w'(s)}{w^2(s)}\sin\phi \right\}$$

$$-\frac{1}{w^3(s)}\cos\phi$$

$$+ky \qquad + kw(s)\cos\phi$$

cancels to 0

must be zero 0

- **The condition that these three coefficients sum to zero is a differential equation for the envelope**

$$\boxed{w''(s) + kw(s) - \frac{1}{w^3(s)} = 0}$$

alternatively

$$\boxed{\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + k\beta^2 = 1}$$

Summary

- **Transverse coordinates**
- **Magnetic rigidity**
- **Fields and force in a quadrupole**
- **Transverse coordinates**
- **Gutter**
- **Transverse ellipse**
- **Alternating gradients**
- **Equation of motion in transverse coordinates**