Lecture 3 - Transverse Optics II

ACCELERATOR PHYSICS

Melbourne

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Lecture 3 - Transverse Optics II
Contents

- Betatron oscillations – More Hill’s Equation.
- Twiss matrix in terms of a, b, g and f.
- Computational methods for lattices
- Smooth approximation
- Beam envelope, emittance.
- Emittance: definition and measurement.
- Physical meaning of Q and beta
- Smooth approximation
- Liouville's theorem.

J.A.I.

Equation of motion in transverse co-ordinates

- Hill's equation (linear-periodic coefficients)

$$\frac{d^2y}{ds^2} + k(s)y = 0$$

  - where $k = -\frac{1}{(B\rho)}\frac{dB_z}{dx}$ at quadrupoles
  - like restoring constant in harmonic motion
- Solution (e.g. Horizontal plane)

$$y = \sqrt{\beta(s)}\sqrt{\varepsilon}\sin\left[\phi(s) + \phi_0\right]$$

- Condition

$$\varphi = \int \frac{ds}{\beta(s)}$$

- Property of machine $\sqrt{\beta(s)}$
- Property of the particle (beam) ε
- Physical meaning (H or V planes)

  Envelope $\sqrt{\varepsilon\beta(s)}$

  Maximum excursions

$$\hat{y} = \sqrt{\varepsilon\beta(s)} \qquad \hat{y}' = \sqrt{\varepsilon / \beta(s)}$$

Twiss Matrix

- **All such linear motion from points 1 to 2 can be described by a matrix like:**

$$\begin{pmatrix} y(s_2) \\ y'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} = \mathbf{M}_{12} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} .$$

- **To find elements first use notation** $w = \sqrt{\beta}$
- **We know** $y = \varepsilon^{1/2} w \ \cos(\varphi + \phi_0)$
- **Differentiate and remember** $\varphi = \frac{1}{\beta} = \frac{1}{w^2}$

$$y' = \varepsilon^{1/2} w' \ \cos(\varphi + \phi_0) - \frac{\varepsilon^{1/2}}{w} \sin(\varphi + \phi_0)$$

- **Trace two rays one starts** $\phi = 0$ **"cosine"**
- **The other starts with** $\phi = \pi / 2$ **"sine"**
- **We just plug in the "c" and "s" expression for displacement an divergence at point 1 and the general solutions at point 2 on LHS**
- **Matrix then yields four simultaneous equations with unknowns : a b c d which can be solved**

Twiss Matrix (continued)

- **Writing** $\phi = \phi_2 - \phi_1$
- **The matrix elements are**

$$M_{12} = \begin{pmatrix} \frac{w_2}{w_1}\cos\varphi - w_2 w_1' \sin\varphi \ , & w_1 w_2 \sin\varphi \\ -\frac{1 + w_1 w_1' w_2 w_2'}{w_1 w_2}\sin\varphi - \left(\frac{w_1'}{w_2} - \frac{w_2'}{w_1}\right)\cos\varphi \ , & \frac{w_1}{w_2}\cos\varphi + w_1 w_2' \sin\varphi \end{pmatrix}$$

- **Above is the general case but to simplify we consider points which are separated by only one PERIOD and for which**

$$w_1 = w_2 = w \ , \ w_1' = w_2' = w' \ , \ \mu = \phi_2 - \phi_1 = 2\pi Q$$

- **The "period" matrix is then**

$$M = \begin{pmatrix} \cos\mu - ww' \sin\mu \ , & w^2 \sin\mu \\ -\frac{1 + w^2 w'^2}{w^2}\sin\mu, & \cos\mu + ww' \sin\mu \end{pmatrix}$$

- **If you have difficulty with the concept of a period just think of a single turn.**

Twiss concluded

$$M = \begin{pmatrix} \cos\mu - ww'\sin\mu\ , & w^2 \sin\mu \\ -\dfrac{1 + w^2 w'^2}{w^2}\sin\mu, & \cos\mu + ww'\sin\mu \end{pmatrix}$$

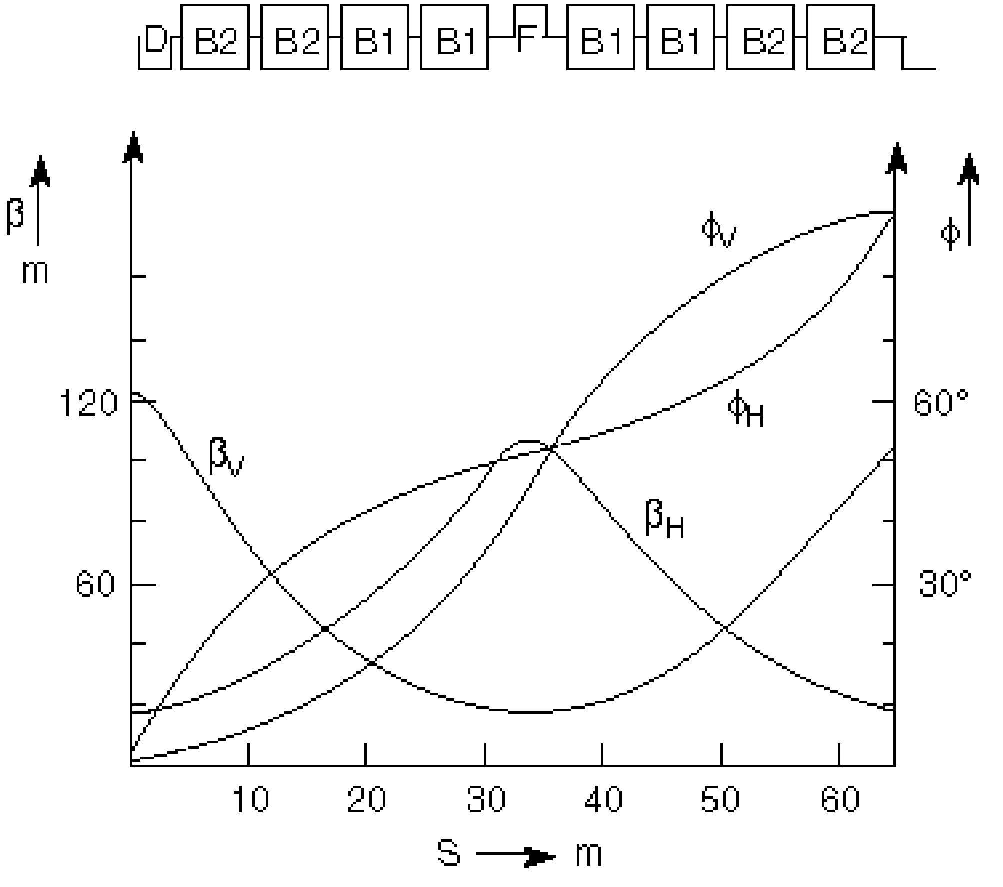
- **Can be simplified if we define the "Twiss" parameters:**

$$\beta = w^2\ ,\ \alpha = -\frac{1}{2}\beta'\ ,\ \gamma = \frac{1+\alpha^2}{\beta}$$

- **Giving the matrix for a ring (or period)**

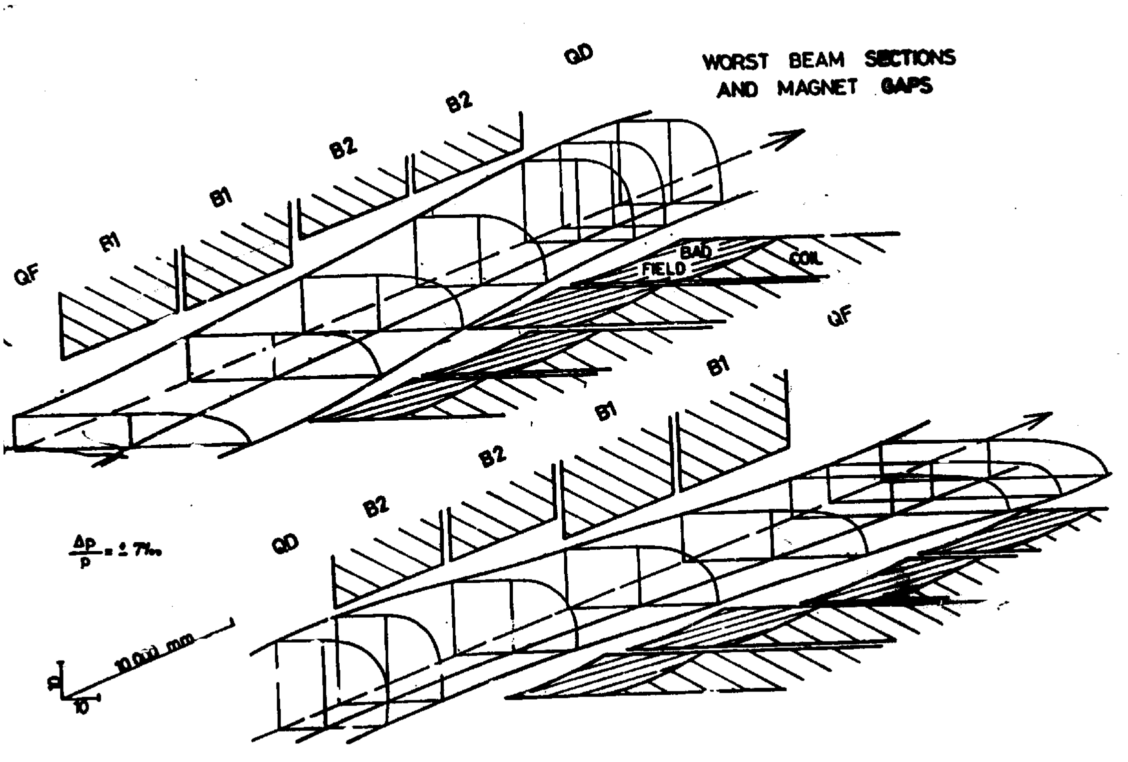
$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu\ , & \beta\sin\mu \\ -\gamma\ \sin\mu, & \cos\mu - \alpha\ \sin\mu \end{pmatrix}$$

The lattice



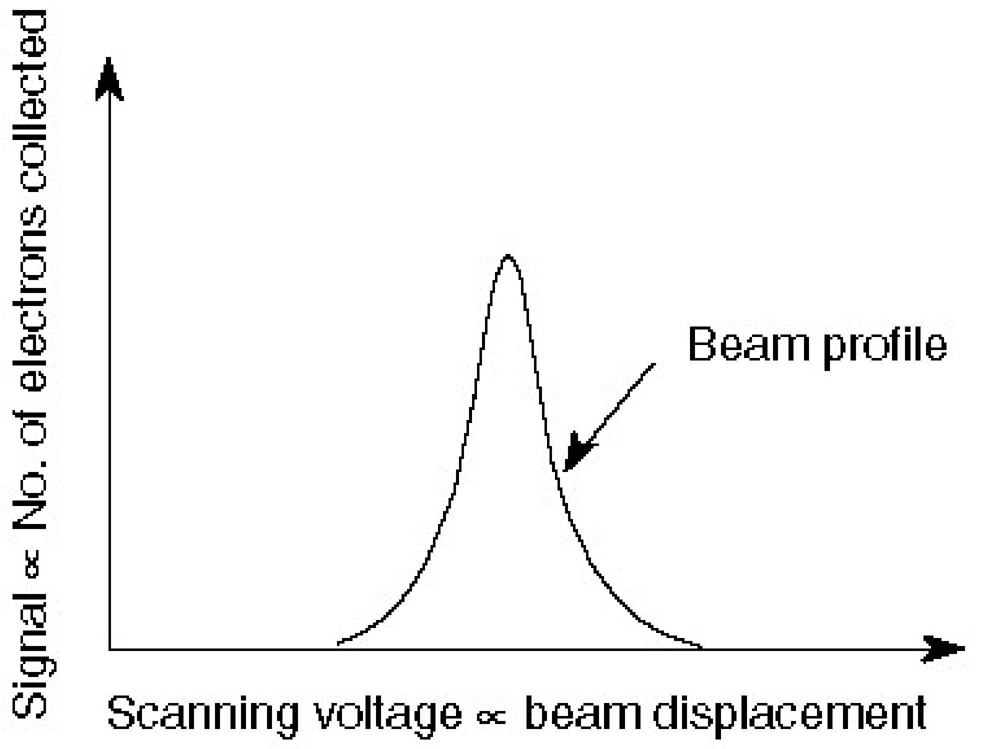
	LENGTH	ANGLE	K(V)	ALPHA(P)	BETA(H)	ALPHA(H)	MUH/2PI	BETA(V)	ALPHA(V)	MUV/2PI	AH/2	AV/2
01	3,085000	0,000000	-,015063	1,386440	104,884855	2,452160	,004571	19,011703	-,520345	,026571	65,715663	9,917560
2	,360000	0,000000	0,000000	1,374653	103,127965	2,428089	,005122	19,395014	-,544408	,029555	64,547513	10,017039
03	6,260000	,008445	0,000000	1,196124	75,348859	2,009521	,016433	28,828710	-,962519	,072198	64,004371	12,212911
4	,400000	0,000000	0,000000	1,186405	73,751941	1,982775	,017287	29,609417	-,989248	,074377	54,751341	12,376828
05	6,260000	,008445	0,000000	1,060742	51,548094	1,564207	,033474	44,610910	-1,407071	,101988	54,174091	15,192432
6	,390000	0,000000	0,000000	1,054559	50,338182	1,538130	,034692	45,718585	-1,433122	,103302	45,428681	15,379447
07	6,260000	,008445	0,000000	,981762	33,701223	1,119863	,058975	66,274961	-1,850527	,121441	44,905056	18,517478
8	,380000	0,000000	0,000000	,978948	32,860011	1,094184	,060793	67,691002	-1,875896	,122344	36,980337	18,713705
09	6,260000	,008445	0,000000	,959017	21,781569	,675586	,098381	93,787676	-2,292753	,134861	36,534921	22,028267
10	2,342700	0,000000	0,000000	,961460	18,983146	,518942	,116758	104,896272	-2,449038	,138621	30,069327	23,295624
11	3,085000	0,000000	,015037	1,034354	18,983068	-,518916	,143368	104,901820	2,447388	,143191	28,349412	23,716525
12	,350000	0,000000	0,000000	1,050730	19,354500	-,542318	,146275	103,198611	2,424067	,143726	28,638028	23,296218
13	6,260000	,008445	0,000000	1,370047	28,764399	-,960879	,189011	75,452122	2,007802	,165027	35,089639	23,106121
14	,380000	0,000000	0,000000	1,391035	29,504322	-,986287	,191088	73,935822	1,982463	,165836	35,546047	19,757412
15	6,260000	,008445	0,000000	1,763219	44,472640	-1,404847	,218731	51,724094	1,565610	,171975	43,750575	19,557880
16	,390000	0,000000	0,000000	1,788053	45,578591	-1,430924	,220109	50,513067	1,539589	,173189	44,298587	16,358398
17	6,260000	,008445	0,000000	2,213103	66,113699	-1,849484	,238298	33,849177	1,122280	,197377	53,470174	16,165762
18	,400000	0,000000	0,000000	2,241952	67,603985	-1,876229	,239251	32,962034	1,095579	,199283	54,079136	13,233307
19	6,260000	,008445	0,000000	2,719868	93,714254	-2,294790	,251780	21,859390	,677943	,230745	63,830251	13,058741
20	2,352700	0,000000	0,000000	2,909420	104,882261	-2,452099	,255558	19,038995	,520847	,255140	67,592709	10,634409
21	3,085000	0,000000	-,015063	2,946010	104,882266	2,452098	,260129	19,038106	-,520546	,281673	68,853088	9,924676
22	,360000	0,000000	0,000000	2,925443	103,125421	2,428027	,260680	19,421551	-,544579	,284663	67,665889	10,023890
23	6,260000	,008445	0,000000	2,594240	75,347037	2,009467	,271992	28,854181	-,962177	,327246	67,105194	12,218305
24	,400000	0,000000	0,000000	2,574765	73,750162	1,982722	,272846	29,634602	-,988874	,329424	57,546939	12,382087
25	6,260000	,008445	0,000000	2,296428	51,546933	1,564162	,289032	44,628208	-1,406185	,356957	56,950187	15,195377
26	,390000	0,000000	0,000000	2,280734	50,337057	1,538085	,290251	45,735180	-1,432204	,358331	47,899567	15,382238
27	6,260000	,008445	0,000000	2,055264	33,700612	1,119525	,314534	66,276862	-1,849098	,376466	47,356928	18,517744
28	,380000	0,000000	0,000000	2,043182	32,859428	1,094117	,316382	67,691805	-1,874435	,377369	39,127022	18,713817
29	6,260000	,008445	0,000000	1,870577	21,781395	,675557	,353941	93,766993	-2,290782	,389888	38,663082	22,025838
30	2,342700	0,000000	0,000000	1,815875	18,983101	,518917	,372318	104,865902	-2,446875	,393648	31,892336	23,292251
31	3,085000	0,000000	,015037	1,873603	18,983178	-,518943	,398928	104,862544	2,447912	,398220	30,027986	23,712598

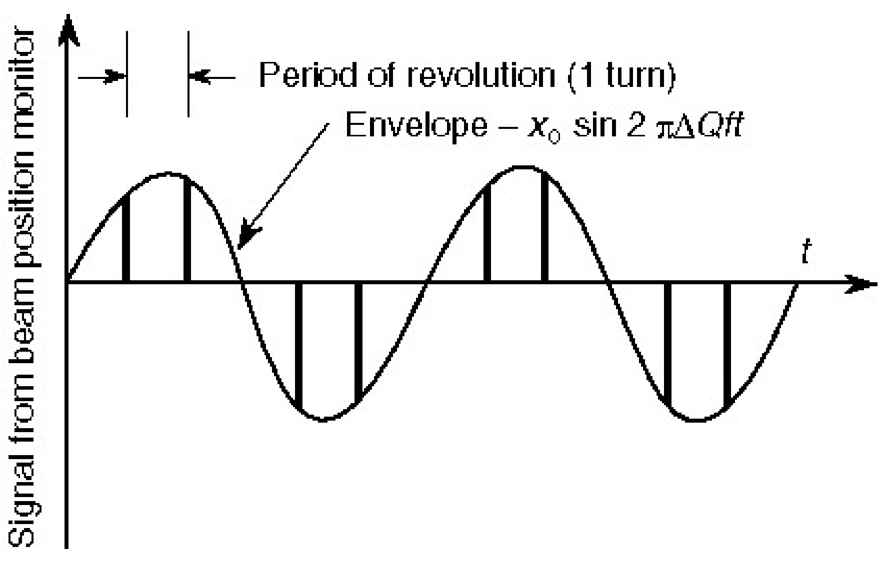
Beam sections



after,pct

Physical meaning of Q and βετα



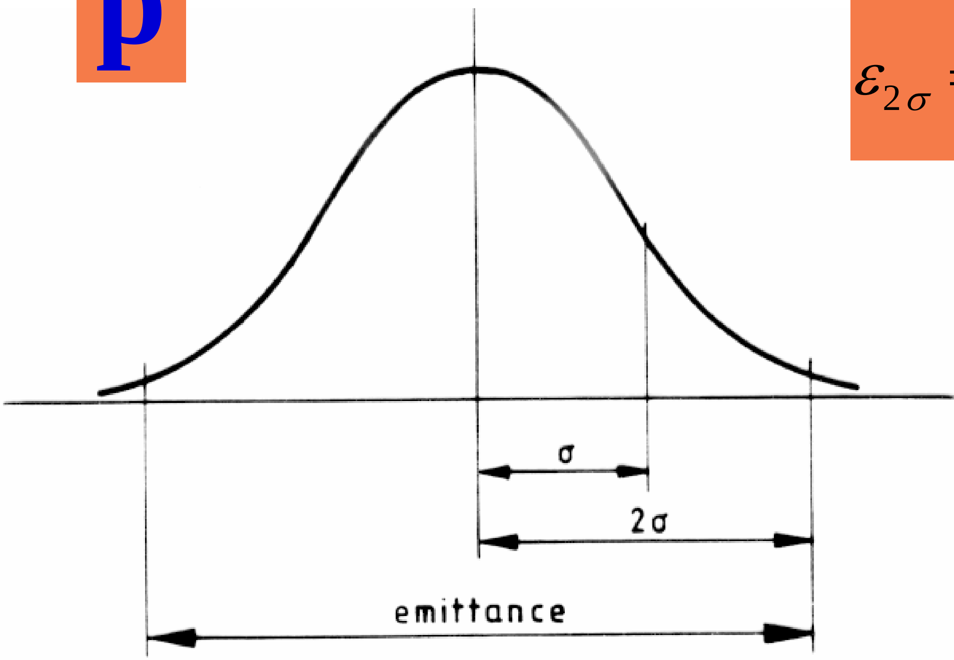


Emittance definitions

$$y = \sqrt{\varepsilon\beta} \qquad \therefore \varepsilon = \frac{y^2}{\beta}$$

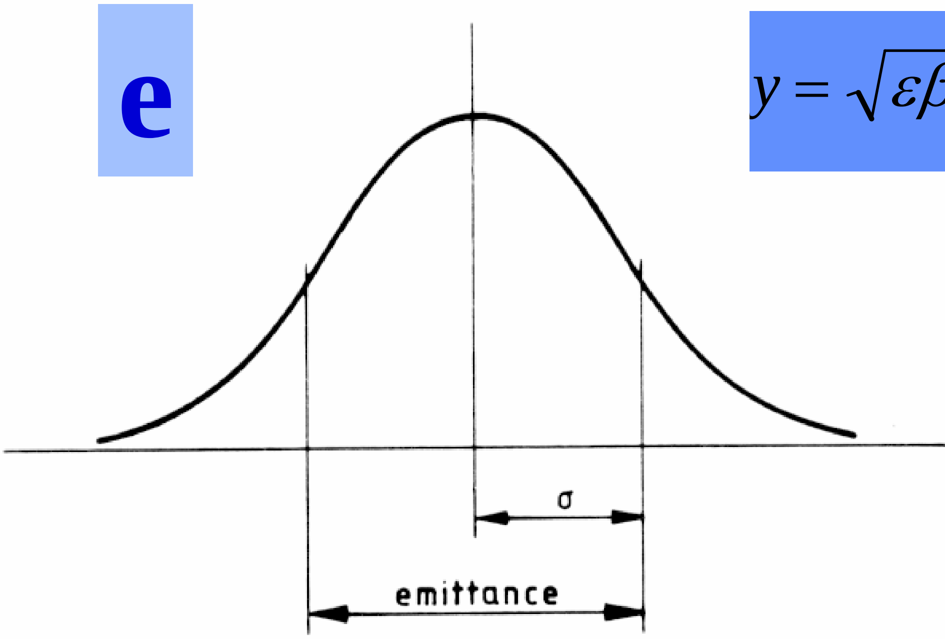
$$\varepsilon_{2\sigma} = \frac{(2\sigma)^2}{\beta} = \frac{4\sigma^2}{\beta}$$

p



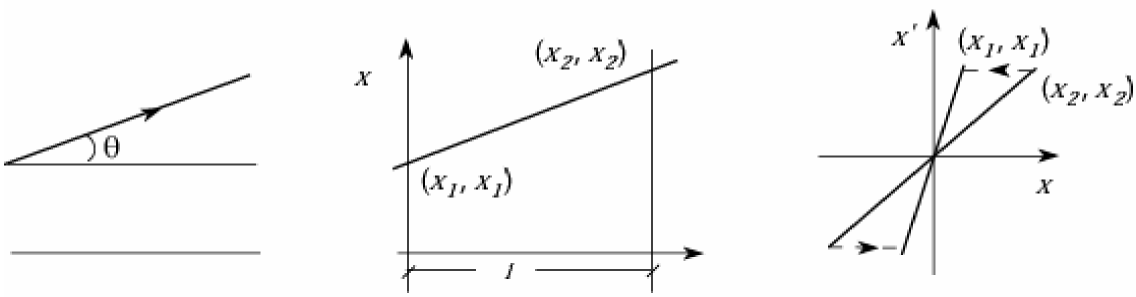
e

$$y = \sqrt{\varepsilon\beta} \qquad \varepsilon_{\sigma} = \frac{\sigma^2}{\beta}$$

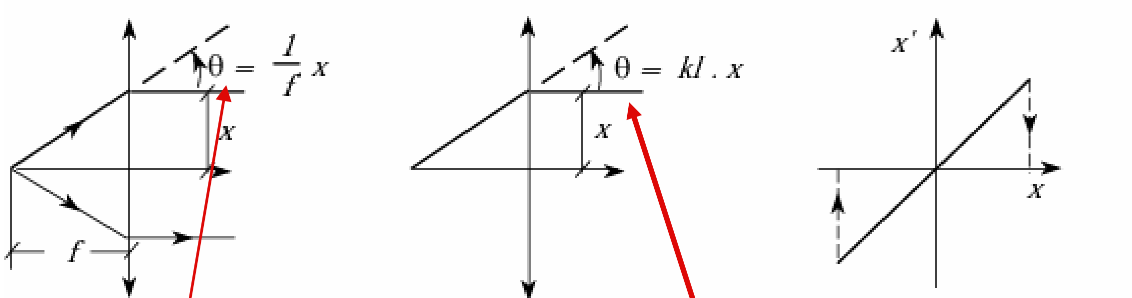


Effect of a drift length and a quadrupole

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$



Drift length



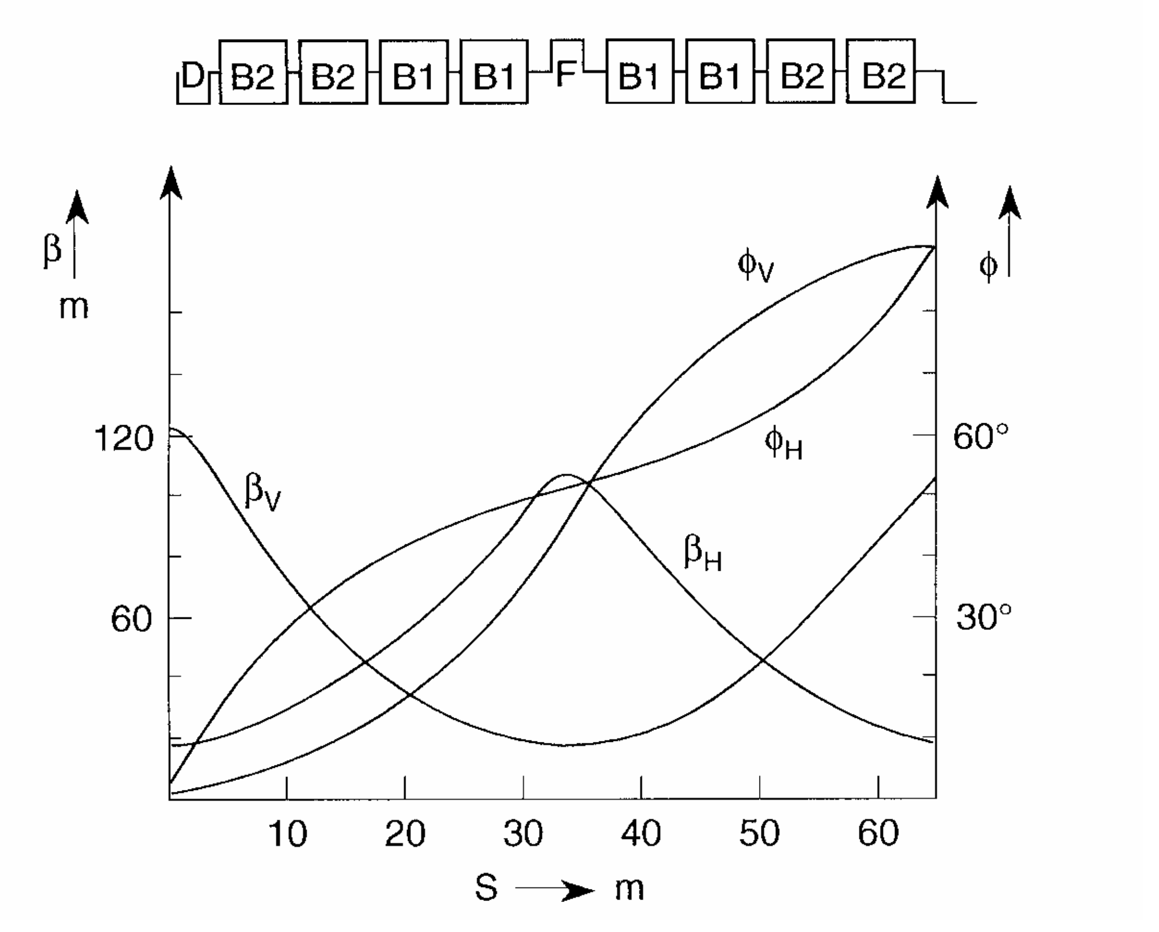
Quadrupole

$$\theta = \frac{1}{f} \cdot x$$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 \,, & 0 \\ -1/f \,, & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 \,, & 0 \\ -kl \,, & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

The lattice (1% of SPS)



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30	2,342700	0,000000	0,000000	1,815875	18,983101	,518917	,372318	104,865902	-2,446875	,393648	31,892336	23,292251
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Calculating the Twiss parameters

THEORY →

COMPUTATION (multiply elements) →

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu\ , & \beta\sin\mu \\ -\gamma\ \sin\mu, & \cos\mu - \alpha\ \ \sin\mu \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Real hard numbers

Solve to get Twiss parameters:

$$\mu = \ \cos^{-1}\left(\frac{Tr\ \ M}{2}\right) = \ \cos^{-1}\left(\frac{a+d}{2}\right)$$

$$\beta = b / \sin\ \ \mu$$

$$\alpha = \frac{a-d}{2\sin\ \ \mu}$$

$$\gamma = -c / \sin\ \ \mu$$

Smooth approximation

$$N\mu = 2\pi Q$$

$$\int \frac{ds}{\beta} = \int d\phi$$

$$\frac{2\pi R}{\bar{\beta}} = 2\pi Q$$

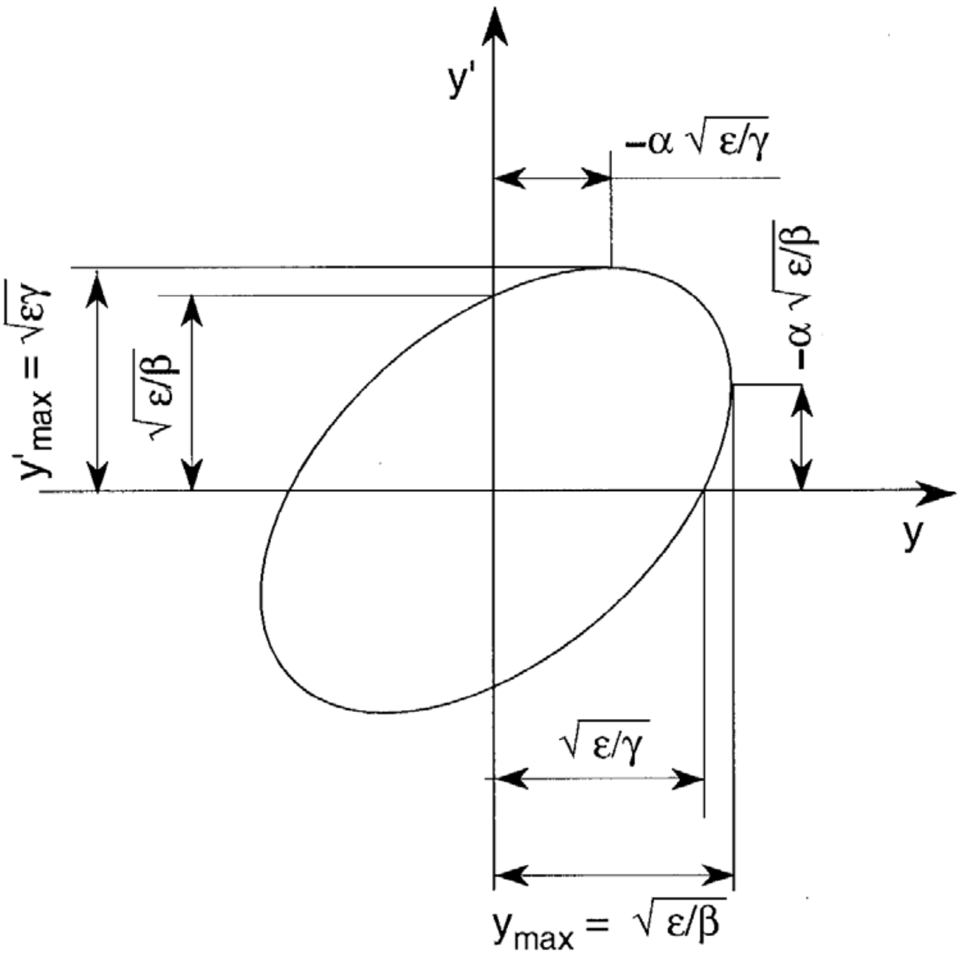
$$\therefore \bar{\beta} = \frac{R}{Q}$$

$$\gamma_{tr} \approx Q$$

$$\frac{1}{{\gamma_{tr}}^2} = \frac{\bar{D}}{R}$$

$$\therefore \bar{D} = \frac{R}{Q^2}$$

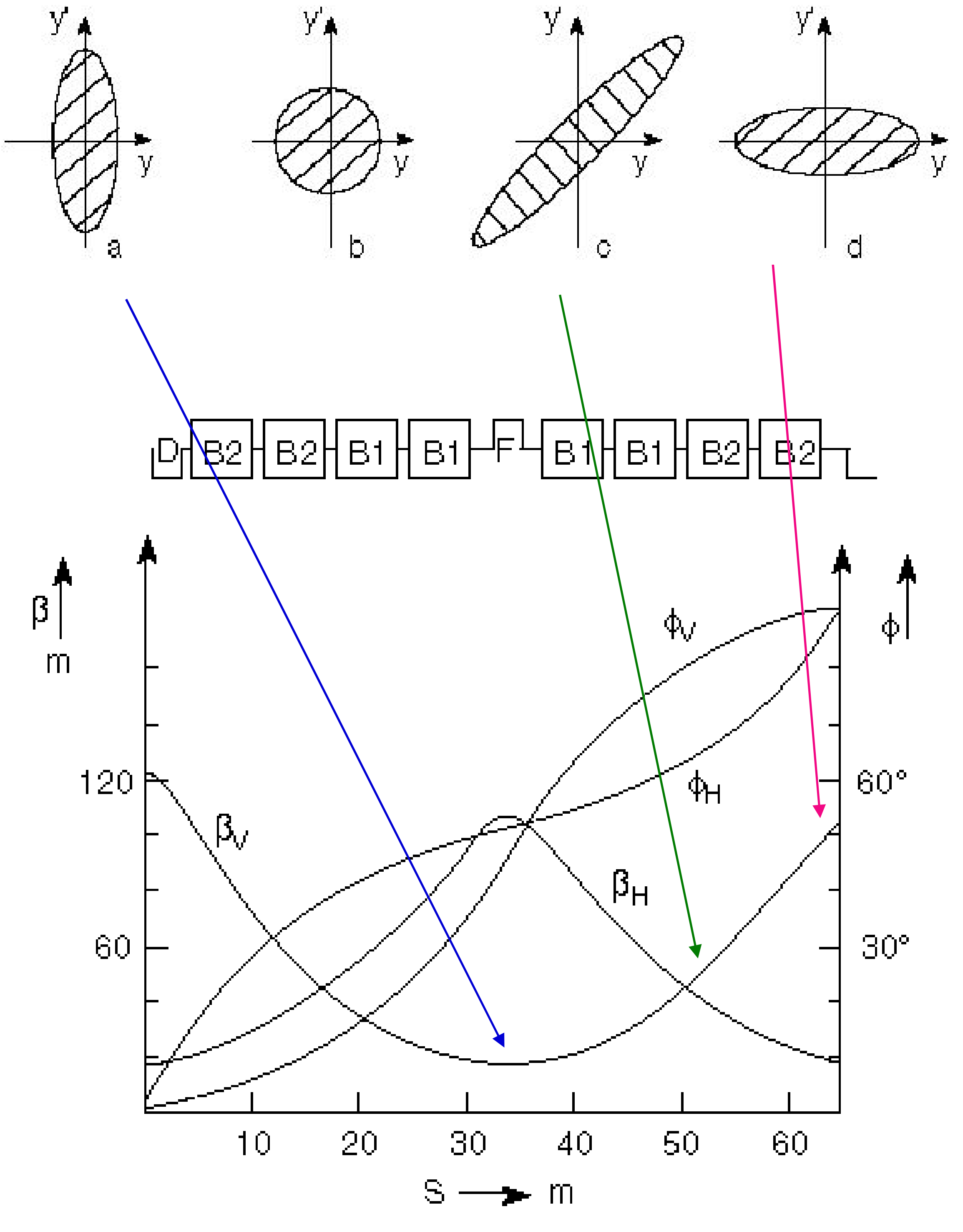
Meaning of Twiss parameters



- **ε is either :**
  - **Emittance of a beam anywhere in the ring**
  - **Courant and Snyder invariant fro one particle anywhere in the ring**

$$\gamma(s)y^2 + 2\alpha(s)yy' + \beta(s)y'^2 = \varepsilon$$

Betatron phase space at various points in a lattice - Liouville

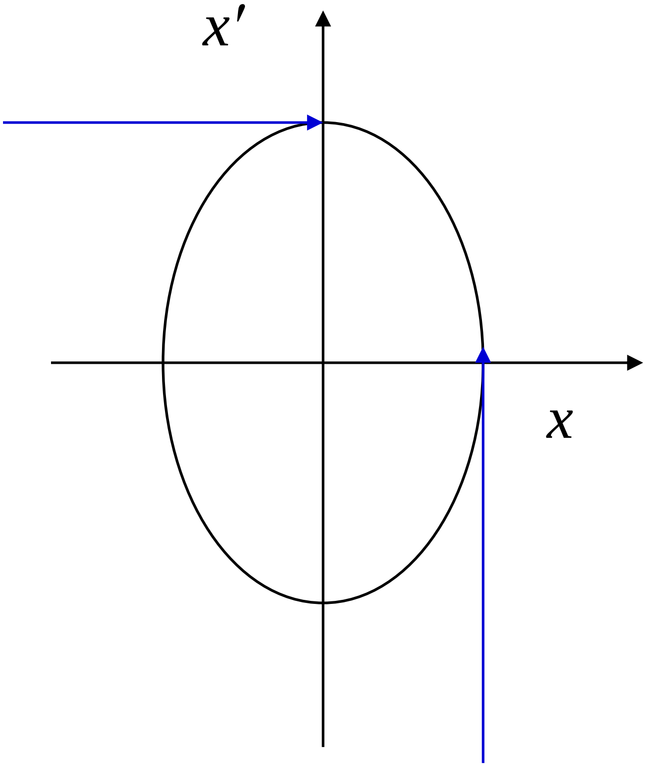


Example of Beam Size Calculation

- **Emittance at 10 GeV/c**

$$\varepsilon = 20\pi\,\text{mm.mrad} = 20\pi \times 10^{-6}\ \text{m.rad}$$

$$\hat{\beta} = 108\ \text{m}$$



$$\sqrt{\varepsilon/\beta} = \sqrt{20.10^{-6}/108}$$

$$= 0.43\sqrt{10^{-6}}$$

$$= 0.43.10^{-3}\ \text{rad}$$

$$= 0.43\ \text{mrad.}$$

$$\sqrt{\varepsilon\beta} = \sqrt{108.20.10^{-6}}$$

$$= 46\sqrt{10^{-6}}$$

$$= 46.10^{-3}\ \text{m}$$

$$= 46\ \text{mm.}$$

Summary

- Betatron oscillations – More Hill’s Equation.
- Twiss matrix in terms of a, b, g and f.
- Computational methods for lattices
- Smooth approximation
- Beam envelope, emittance.
- Emittance: definition and measurement.
- Physical meaning of Q and beta
- Smooth approximation
- Liouville's theorem.